JOURNAL OF COMPUTATIONAL PHYSICS 46, 43–53 (1982)

On the Kantorovich Technique Applied to the Tidal Equations in Elongated Lakes

GABRIEL RAGGIO*

Laboratory of Hydraulics, Hydrology, and Glaciology
The Federal Institute of Technology, Zurich, Switzerland

Received June 24, 1981

Remarks are made regarding numerical properties of a system of ordinary differential equations that results from the application of the Kantorovich technique to the tidal equations. It is shown that shape functions can be specified by a "problem-oriented inner product" leading to an orthogonal-shape function set. Such a choice may lead to substantial computational savings.

1. INTRODUCTION

The traditional approach to the analysis of free- and forced-seiche motions in homogeneous water bodies, especially lakes, was the "channel" approximation (for a review see Defant [3]) but with this simple model the slowly rotating waves, which appear in lakes and can be traced to the Earth's rotation, cannot be reproduced. A first attempt to account for the rotation of the Earth in seiches in lakes was made by Defant [2] for Lake Michigan using the idea of A. Defant. The idea is to assume that the seiche motion exhibits Kelvin-wave character; thus, to the solution of the channel approximation a transverse slope of the free surface is superimposed, obtained from the geostrophic balance. This well describes the amphidromic system of the lowest mode in long lakes as proven by Rao [14] for long rectangular basins. Since the Kelvin–wave hypothesis is not always justified, especially for rounded lakes and higher modes the channel approach was abandoned and solution procedures for the two-dimensional linear-tidal equations were searched for. For natural basins finite-difference (Platzman [9], Rao and Schwab [15]) or finite-element (Hamblin [7]) methods were used.

Since many lakes are elongated the most likely excited modes are the lowest ones having predominantly longitudinal character. Thus, the success of the channel approximation is because "... it replaces a difficult two-dimensional problem by a highly tractable one-dimensional problem" (Platzman [9]). Discussions on the difficulties of the two-dimensional equations are given by Platzman [9] and

* Financially supported by the Swiss National Science Foundation through the national program, "Basic Problems of Swiss Water Budget," Contract No. 4.006.0.76.02.

0021-9991/82/040043–11$02.00/0

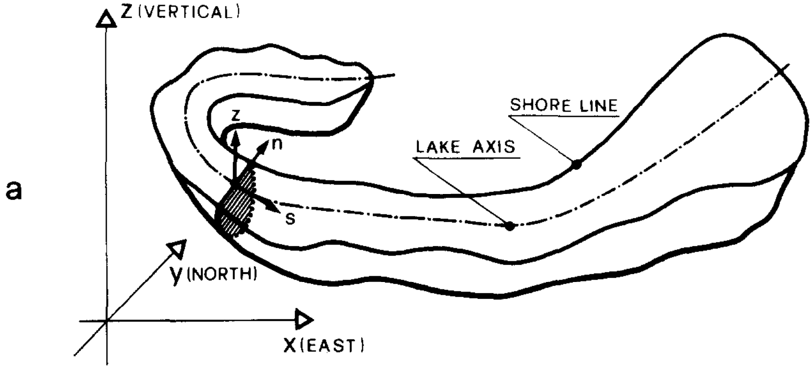

FIG. 1. Elongated curved lake embedded in three-dimensional space. Equations in the text refer to the curvi-linear coordinate system (s, n, z) in the "long" and transverse directions of the lake.

Hamblin [6]. The approach of this paper is to use the Kantorovich method (Kantorovich and Krylov [8]) to reduce partial differential equations to ordinary differential equations. This method is outlined for the problem at hand in Raggio [10]), Raggio and Hutter [11]; analytical tests of the equations and results for free and forced oscillations in a natural lake are given by Raggio and Hutter [11, 12].

The numerical analysis of the above-mentioned extended-channel model are dealt with. Periodic solutions in time are assumed. The linear-tidal equations are formulated in a plane right-handed orthogonal curvilinear-coordinate system along the "axis" of the lake, see Fig. 1; subsequently, the field variables, which are functions of the longitudinal and transverse coordinates, are expanded with shape functions in the transverse coordinate only. Thus, any field variable x may be expressed as $x = \boldsymbol{\phi} \cdot \mathbf{x}$, where $\boldsymbol{\phi} = \boldsymbol{\phi}(n)$ is the vector of the shape functions and $\mathbf{x} = \mathbf{x}(s)$ the unknown amplitude vector of dimension N; s and n are longitudinal and transverse coordinates, respectively. These expansions are used in weighted-residual

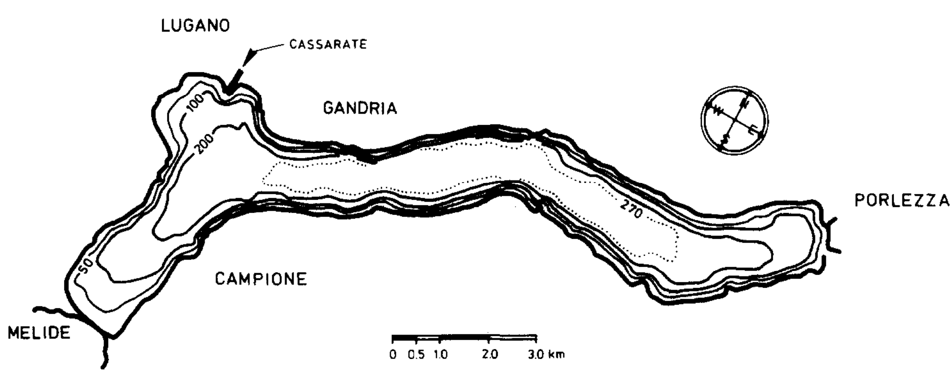

FIG. 2. Top view of the eastern basin of Lake of Lugano with bathymetry.

expressions in which a Galerkin procedure is chosen. What emerges is a set of ordinary differential equations, in our case a two-point boundary value problem. The two boundary points are the extreme ends of the channel axis. In the above-mentioned Kantorovich technique the shape functions need not necessarily fulfill the lateral-boundary conditions since a slip condition was assumed, which is a natural-boundary condition of the original problem.

The analysis was carried out for the eastern basin of the Lake of Lugano (at the Swiss–Italian border), and here all results relate to this lake; for a top view with bathymetry see Fig. 2.

2. Basic Equation

The governing equations are a set of mixed-ordinary differential and algebraic equations. The first arise by applying the Kantorovich technique to the continuity equation and longitudinal-momentum balance, the latter by applying this technique to the momentum balance in transverse direction. Introducing as amplitude vectors the longitudinal velocity v_s, the transverse velocity v_n, and the surface elevation ξ, the equations read

$$\mathbf{D}\frac{d\mathbf{y}}{ds} + \mathbf{B}\mathbf{y} + \mathbf{C}\mathbf{v}_n = \mathbf{l}_s, \qquad \mathbf{E}\mathbf{y} + \mathbf{A}\mathbf{v}_n = \mathbf{l}_n, \tag{2.1}$$

in which

$$\mathbf{y} = (\mathbf{v}_s, \boldsymbol{\xi})^{\mathrm{T}} \tag{2.2}$$

and

$$\begin{gathered}\mathbf{B} = \begin{pmatrix} {}_1\mathbf{R} + i\omega\,{}_1\mathbf{C} & \mathbf{0} \\ \partial\,{}_0\mathbf{C}/\partial s & i\omega\,{}_1\mathbf{Z} \end{pmatrix}, \qquad \mathbf{C} = \begin{pmatrix} f\cdot{}_1\mathbf{C} \\ {}_1\mathbf{C}_\phi^{\mathrm{T}} \end{pmatrix}, \\ \mathbf{E} = (f\,{}_1\mathbf{C}\;\; g\,{}_1\mathbf{C}_\phi), \qquad \mathbf{A} = ({}_1\mathbf{R} + i\omega\,{}_1\mathbf{C}),\end{gathered} \tag{2.3}$$

where ω is the frequency of oscillation, f is the Coriolis parameter and g the acceleration due to gravity. Further, $\mathbf{l}_s$ and $\mathbf{l}_n$ are two forcing vectors which take into account wind drag and atmospheric-pressure differences and the matrix ${}_1\mathbf{R}$ considers bottom friction. The submatrices listed in (2.3) are cross-sectional constants; when Q is the cross section, H the depth of the lake and J the Jacobian of the curvilinear-coordinate system, then

$$\begin{gathered}{}_m\mathbf{C} = \iint_Q HJ^m \boldsymbol{\phi}\cdot\boldsymbol{\phi}^{\mathrm{T}}\,dn, \qquad {}_1\mathbf{C}_\phi = \iint_Q HJ\boldsymbol{\phi}\frac{\partial\boldsymbol{\phi}^{\mathrm{T}}}{\partial n}\,dn, \\ {}_1\mathbf{Z} = \int_{B-}^{B+} J\boldsymbol{\phi}\cdot\boldsymbol{\phi}^{\mathrm{T}}\,dn, \\ {}_1\mathbf{R} = \int_{B-}^{B+} \frac{J}{R}\boldsymbol{\phi}\cdot\boldsymbol{\phi}^{\mathrm{T}}\,l(s,n)\,dn, \qquad l(s,n) = \left[1 + \left(\frac{\partial H}{\partial n}\right)^2 + \left(\frac{1}{J}\frac{\partial H}{\partial s}\right)\right]^{1/2},\end{gathered} \tag{2.4}$$

in which B^+ and B^- are the 2 shore lines and R a friction coefficient. It relates shear traction at the bottom to the tangential-sliding velocity, $v = R\tau$, and need not be constant. Further, these submatrices are $(N \times N)$ matrices and, thus, (2.1) is a system of $2N$ differential equations and N algebraic equations.

Boundary conditions for a lake, neglecting inflow and outflow, are that no flux occurs at the two endpoints which can be expressed as

$$(1\ \ \mathbf{0})\mathbf{y} = \mathbf{H}\mathbf{y} = 0 \qquad \text{at} \quad s = 0,\ L, \tag{2.5}$$

where L is the length of the lake along the axis.

Since $\mathbf{A}$ is nonsingular, if $\omega \neq 0$, and because ${}_o\mathbf{C}$ is positive definite (2.1) with boundary conditions, (2.4) may be reduced to a standard two-point boundary value problem in the form

$$\mathbf{y}'(s) = \mathbf{F}(s, \omega)\, \mathbf{y}(s) + \mathbf{a}(s, \omega) \tag{2.6a}$$

$$\mathbf{H}\mathbf{y}(0) = \mathbf{0}, \qquad \mathbf{H}\mathbf{y}(L) = \mathbf{0}. \tag{2.6b}$$

To determine the natural frequencies of oscillation ω of a basin the friction terms in (2.3) and $\mathbf{a}$ in (2.6a) must be dropped.

3. Integration Method

As a method of integration for the two-point boundary value problem (2.6) the initial-value approach was chosen. In this method linear-independent solutions fulfilling the boundary condition at the initial boundary are linearly combined and the second-boundary condition determines the free constants of superposition, see Scott and Watts [16].

The local behaviour of the differential equation (2.6) at a point s_o for a fixed frequency of oscillation ω_0 may be characterized by the eigenvalues λ_i, $i = 1, \ldots, 2N$ of the matrix $\mathbf{F}(s_o, \omega_o)$. (These λ_i should not be confused with the eigenvalues ω_j which are the natural frequencies of oscillation). If we consider $\mathbf{F}$ to be constant in an interval of s near s_o assuming that the λ_i are all distinct, the general solution is given by

$$\mathbf{y}(s) = \sum_{i=1}^{2N} \alpha_i e^{\lambda_i (s - s_o)} \cdot \boldsymbol{\beta}_i, \tag{3.1}$$

where the α_i are constants determined essentially by the values of $\mathbf{y}$ at s_o and the eigenvalues λ_i, and corresponding eigenvectors $\boldsymbol{\beta}_i$ fulfill the equation

$$\mathbf{F}(s_o, \omega_o) \cdot \boldsymbol{\beta}_i = \lambda_i \boldsymbol{\beta}_i. \tag{3.2}$$

Arranging the eigenvalues in ascending order according to their real parts, $\mathrm{Re}(\lambda_1) < \cdots < \mathrm{Re}(\lambda_{2N})$, the spectral width is then

$$d = \mathrm{Re}(\lambda_{2N}) - \mathrm{Re}(\lambda_1). \tag{3.3}$$

For practical frequency ranges the spectral width of the matrix $\mathbf{F}$ increases with increasing number of shape functions per variable. In other words, the real parts of the eigenvalues of $\mathbf{F}$ lie farther and farther apart with increasing number of shape functions.

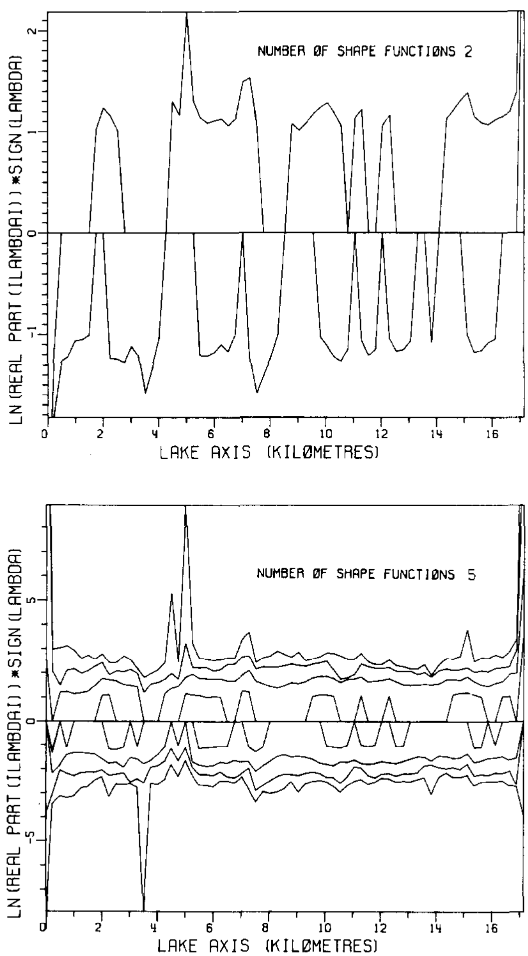

FIG. 3. Real parts of eigenvalues $[\mathrm{km}^{-1}]$ of the differential equation along the axis of the lake for the free-oscillation case. The frequency corresponds to the fourth gravity mode ($\omega = 0.280018$ Hz) of the eastern basin of Lake of Lugano. Figures 3a and b display the real parts using 2 and 5 Cauchy terms as shape functions.

As a consequence of the large-spectral width of the matrix **F** the linearly independent functions become numerically linearly dependent as integration proceeds (see Eq. 3.2). It is overcome by dividing the integration interval into subintervals in which the superposition solutions may be maintained numerically linearly independent. The software package SUPORT of Sandia Laboratories, developed by Scott, Watts, and Lord (see, e.g., Scott and Watts [16], and Watts *et al.* [17]) was used. It was primarily designed to solve by an initial-value integration process, unstable-linear two-point boundary value problems and problems strongly sensitive to initial conditions. It overcomes the difficulty of independent-superposition solutions becoming linearly or almost-linearly dependent by an orthonormalization procedure. The SUPORT package allows for several alternatives for its implementation. For our application these are discussed by Raggio [10].

The eigenvalues λ_i represent the different length scales which are included in the model. This is displayed in Figs. 3a and b for a frequency near the fourth eigenmode $\omega = 0.280018$ Hz, (for details, see Ragio [10]). The abscissa represents the axis of the lake measured in km from its soutwestern to its northeastern end. The natural logarithm of the absolute values of the real parts times the sign of the real parts are displayed on the ordinate; when its absolute value is smaller than unity, zero is shown. The physical interpretation of the general growth of the spectral width is that, as the number of shape functions increases shorter and shorter wavelength scales are included in the model. These are represented by the large eigenvalues which are essentially wavenumbers. The peaks in Fig. 3b can be correlated with the variation of the cross-sectional area along the lake axis. This is so because a reduction in the cross section will cause retarding of waves and, thus, a strong increase of the real parts of the eigenvalues and vice versa.

To conclude this section we shall show in Table I the number of orthonormalization points determined by the SUPORT package when integrating from Melide

TABLE I

Number of shape functions	Number of orthonormalization points	Number of orthonormalization points between 5 and 5.5 km from Melide
1	—	—
2	3	—
3	7	1
4	16	7
5	87	72

Note. Number of shape functions and the resulting number of orthonormalization points for the fourth mode of oscillation ($\omega = 0.280018$ Hz) using the Default parameter for the orthonormalization test in the SUPORT package. Distances are measured along the axis of the lake from Melide Dam.

to Porlezza. It is clearly seen that the number of orthonormalization points increases drastically with the number of shape functions. The table also shows the clustering of these points in the domain with large-real parts of eigenvalues for **F**, see Fig. 3b.

4. Choice of Shape Functions

In the Kantorovich method, shape functions may be selected from physically meaningful assumptions of the particular phenomena one would like to simulate. Another criterion may be to reduce the computational effort by selecting shape functions from special orthogonal families. Such families are generated by inner products of the form

$$(P_i, P_j) = \int_a^b w(n)\, P_i(n)\, P_j(n)\, dn = \|P_i\|, \qquad i = j, \\ = 0, \qquad i \neq j \tag{4.1}$$

in which P_i and P_j are two members of the family and w is a weighting function. A sufficient condition for (4.1) to be an inner product is that a and b are finite and $w > 0$. Among the computationally efficient families are the polynomials, and in the following these will be used and some of the results may be transferred to other nonpolynomial families. Orthogonal polynomials can be constructed with three-term recursive formulas.

The idea is to select shape functions $\boldsymbol{\phi}$ which diagonalize a particular submatrix in (2.1) and, thus, are orthogonal with respect to that particular inner product. The criterion for the selection of the appropriate family will be the least number of elementary operations to evaluate the right-hand side of (2.6a) in the initial-value approach with a Runge-Kutta integrator at each point along the axis. It turns out that the most convenient choice is to make ${}_1\mathbf{C}$ diagonal or the unit matrix. From (2.4) the weighting function w is the depth of the cross section weighted with the Jacobian, $w = HJ$, which is always greater or equal to zero and thus admissible. With this choice ${}_1\mathbf{C}_\phi$ is upper triangular with zero-diagonal entries, since

$$({}_1\mathbf{C}_\phi)_{ij} = (P_i, P_{j,n}),$$

and P_j is a polynomial of order j, thus $P_{j,n}$ is of order $(j-1)$ which can be expressed as a linear combination of the first j members of the family,

$$P_{j,n} = \sum_{k=0}^{j} a_k P_k$$

and, therefore,

$$(P_i, P_{j,n}) = \sum_{k=0}^{j=1} a_k (P_i, P_k) = \sum_{k=0}^{j=1} a_k \|P_i\|\, \delta_{ki}$$

$$= 0, \qquad i \geqslant j,$$
$$= a_i \|P_i\|, \qquad i < j.$$

This proves triangularity with zero diagonal elements. Further, if these are chosen as orthogonal polynomials ${}_0\mathbf{C}$ will be nearly the unit matrix, since it has H rather than HJ as weighting factor in the inner product. Thus, $d({}_0\mathbf{C})/ds$ is nearly the zero matrix, which causes a reduction of the number of orthonormalization points in the integration process. This is because the spectral width of the eigenvalues of $\mathbf{F}$ will show reduced peaks along the channel axis.

For the explicit calculation of the above-mentioned elementary operations one must take into consideration the superposition method as implemented in SUPORT. At each point along the axis where the integration requires evaluation of the differential equation, $(N + 1)$ computations of $\mathbf{F} \cdot \mathbf{y}$ and one computation of $\mathbf{a}$ must be done. In Table II the number of elementary operations are shown. In the first row the operations required for general shape functions with no special orthogonal properties are given. The subsequent rows contain the numbers of operations needed for orthogonal and orthonormal functions which make ${}_1\mathbf{C}$ a diagonal and unit matrix, respectively. The cases of free and forced oscillation are given in the first and second column, respectively. This proves that using these orthogonal-shape functions reduces the number of elementary operations by, roughly, a factor of two. Further, not only have fewer coefficients of the ordinary differential equations been calculated and stored but also fewer coefficients must be interpolated during the integration. For problems with analytic solutions orthogonal functions are more accurate than shape functions with compact support (these lead to banded submatrices and thus have computational advantages as well), since they are higher order representations.

In the following the computational effort required to compute the response of the Lake of Lugano in the frequency range with wind forcing is presented. Three shape functions were used for a Cauchy series (these are simple powers in the transverse coordinate and represent families with no special orthogonal properties), orthogonal and orthonormal polynomials. The sampled frequency range was in the range of the first few seiche modes. As the frequency falls below the first eigenfrequency the system of ordinary differential equations becomes stiffer. Thus for this example, for

TABLE II

Shape function	Free oscillation	Forced oscillation
general	$122/6 * N^3 + 12 * N^2$	$185/6 * N^3 + 59 * N^2 + 16 * N$
orthogonal-${}_1\mathbf{C}$	$61/6 * N^3 + 18 * N^2$	$95/6 * N^3 + 37 * N^2 + 18 * N$
orthonormal-${}_1\mathbf{C}$	$61/6 * N^3 + 12 * N^2$	$71/6 * N^3 + 27 * N^2 + 10 * N$

Note. Elementary operations required at each point along the axis in the integration process for the system (2.6a) in the free- and forced-oscillation cases (for general, orthogonal-${}_1\mathbf{C}$ and orthonormal-${}_1\mathbf{C}$ shape functions).

the frequency range between 0.3×10^{-1} and 0.3×10^{-3} (Hz) the Runga–Kutta integrator of SUPORT was used and for the range 0.3×10^{-3} to 0.3×10^{-4} (Hz) the Adams code of SUPORT was used. The integration was carried out with a relative and absolute tolerance of 5×10^{-2}. The columns of Tables III and IV give the computational requirements for the different shape functions. The first row shows the relative cost of the calculations in the frequency ranges. The following rows have pairs of numbers; the first corresponds to the lower and the second to the upper frequency limit of the frequency ranges in each table. Thus, the second number of the pairs given in Table III may be compared with the first number of the pairs given in Table IV. The second row gives the processing time in seconds, the third row the number of orthonormalization points, the fifth row the number of function evaluations with different arguments for the independent variable (the number of times that coefficients of the differential equations have to be computed and the

TABLE III

Shape function	Orthonormal polynomials	Orthogonal polynomials	Cauchy series
relative cost	0.15	0.30	0.54
processing time (s)	(6,8)	(8,11)	(17,20)
orthonormalization points	(8,24)	(9,26)	(9,25)
integral steps	(31,54)	(35,59)	(43,55)
function evaluations	(217,340)	(261,365)	(339,396)
function calls	(900,1454)	(1080,1560)	(1396,1652)

Note. Computational requirements using orthonormal and orthogonal polynomials and Cauchy series as shape functions in the frequency range between 0.3×10^{-1} and 0.3×10^{-3} (Hz) using the Runge-Kutta integrator.

TABLE IV

Shape function	Orthonormal polynomials	Orthogonal polynomials	Cauchy series
relative cost	0.42	0.67	1.0
processing time (s)	(13,25)	(20,38)	(27,57)
orthonormalization points	(26, 7)	(30,10)	(28, 9)
integral steps	(281,557)	(313,602)	(277,601)
function evaluations	(285,582)	(317,628)	(290,628)
function calls	(2364,4584)	(2636,4960)	(2364,4954)

Note. Computational requirements using orthonormal and orthogonal polynomials and Cauchy series as shape functions in the frequency range between 0.3×10^{-3} and 0.3×10^{-4} (Hz) using the Adams integrator.

function itself has to be evaluated), and finally the sixth row gives the number of function evaluations (referred to as "function calls" in the tables) required after the coefficients of the equations have been interpolated and the LU-decompositions have been made.

Although the orthonormal polynomials are the most efficient, for a fixed-integration tolerance, inspection of the results from these shape functions shows that in some parts, especially the shallow regions of the lake, the solution seemed to be incorrect (i.e., very large values for surface elevation amplitude). This is so because for orthonormal shape functions integration should be carried out with higher accuracy, since the solution $\mathbf{y}$ of the ordinary differential equations (2.6) must be less smooth than when "nonnormalized" shape functions are used. The basis for this is as follows (the argument assumes that the physical solution of a problem in a channel of nonconstant cross section is smooth, i.e., smooth surface elevation in a lake. In a channel-type model the physical solution is computed by the scalar product of the shape functions $\boldsymbol{\phi}$ and the solutions $\mathbf{y}$ of the differential equations): If the shape functions are the same for all cross sections, the solution $\mathbf{y}$ must be smooth if the physical solution is smooth. Normalizing the shape functions (as has been done with the orthonormal polynomials for ${}_1\mathbf{C}$) in a channel of nonconstant cross section, transforms the channel into a canal of constant cross section. Thus, if the physical solution is smooth, the solutions of the differential equations will be less smooth since the shape function may have quite different magnitudes along the axis of the channel-like lake.

The results for Cauchy series and orthogonal polynomials were very similar but, as seen from Tables III and IV, orthogonal polynomials are far more convenient because of the reduction in computational requirements.

5. Closing Remarks

In this article numerical properties of a set of ordinary differential equations have been discussed which evolves from the application of the Kantorovich technique to the spatially two-dimensional tidal equations. The purpose was to demonstrate that in the application of the Kantorovich method, shape functions can be determined with the aim to minimize computational effort. This led to shape functions of polynomial families which are orthogonal with respect to a problem-oriented inner product. It was shown that by this choice computational effort can be reduced in comparison to shape functions with no special orthogonality properties by a factor of, roughly, two.

Since the Kantorovich method is used in many problems of slender bodies (e.g., rod and jet theories, see Antman [1], Green *et al.* [4, 5]) and shape functions are often inconsiderately chosen this observation may be of use.

ACKNOWLEDGMENTS

I thank my advisor Dr. K. Hutter for his guidance while performing this work and Dr. A. Friedli for his advice with the SUPORT package.

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